2.2: Equilibrium Solutions and Stability

The goal of this section is not to solve differential equations explicitly, but rather to extract qualitative information about the general properties of its solutions when it is difficult or impossible to solve it explicitly.

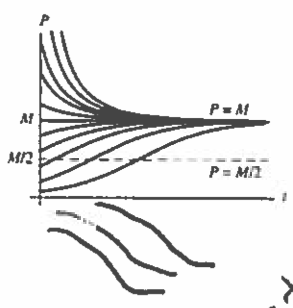
Definition 1.

(a) A differential equation of the form $\frac{dx}{dt} = f(x)$ is called an **autonomous** first-order differential equation.

(b) Similar to Calculus 1, the places where $\frac{dx}{dt} = f(x) = 0$ are called the **critical points** of the autonomous differential equation.

(c) By finding these critical points we can sometimes find **equilibrium solutions** (constant solutions) to the autonomous differential equation.

Example 1.



The slope field for the logistic equation

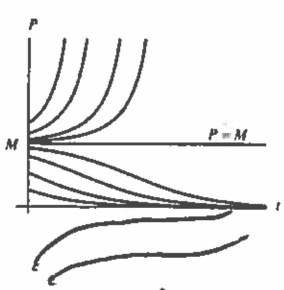
$$\frac{dx}{dt} = kx(M - x)$$

is shown to the left. We can use this to get the **phase diagram** which is a helpful tool in finding stable and unstable critical points.

$x' < 0$ $\quad$ $x' > 0$ $\quad$ $x' < 0$

(unstable) $x = 0$ $\qquad$ $x = M$ (stable)

Example 2.



The slope field for the explosion/extinction equation

$$\frac{dx}{dt} = kx(x - M)$$

is shown to the left. We can use this to get the **phase diagram** which is a helpful tool in finding stable and unstable critical points.

$x' > 0$ $\quad$ $x' < 0$ $\quad$ $x' > 0$

$x = 0$ $\qquad$ $x = M$ (unstable)

(stable

Definition 2. The two phase diagrams we have created for Examples 1 and 2 clearly illustrate the concept of stability. By definition a critical point $x = c$ is **stable** if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x_0 - c| < \delta \quad \text{implies that} \quad |x(t) - c| < \epsilon$$

for all $t > 0$. A critical point is **unstable** if it is not stable.

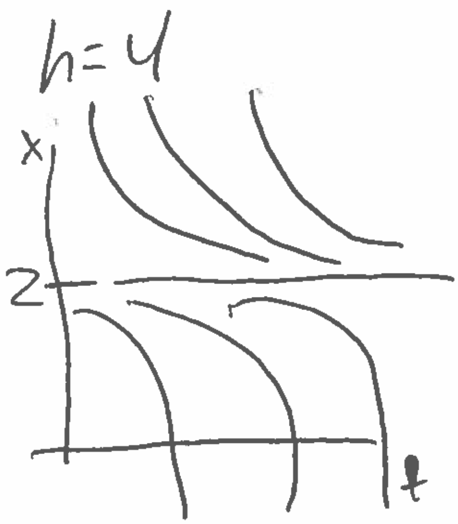
Example 3. Consider the logistic equation with an additional harvesting term

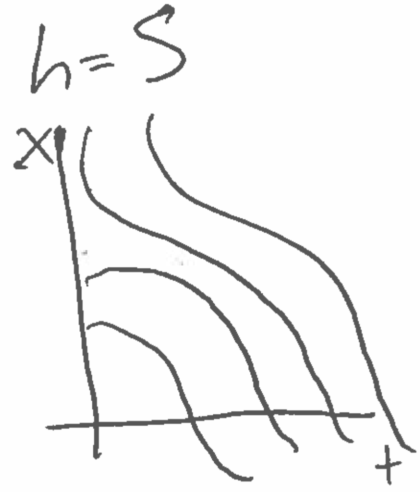
$$\frac{dx}{dt} = kx(M - x) - h.$$

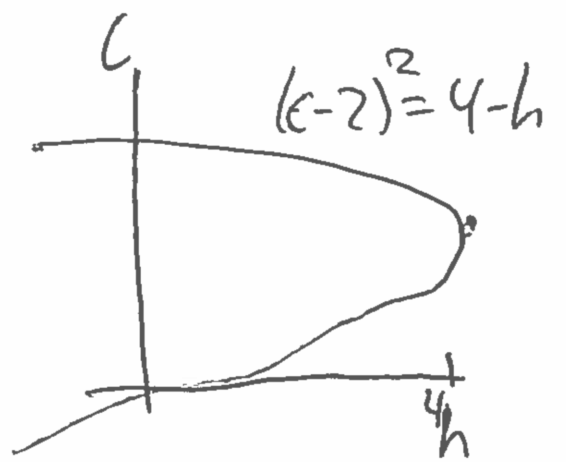
Notice that we have a three-parameter family of equations. Let us fix $k = 1$ and $M = 4$(hundred) to get

$$\frac{dx}{dt} = x(4 - x) - h.$$

Now we have a one-parameter family of equations. So let us consider what happens to the critical points as h varies.







Crit. pts: $c = 2 \pm \sqrt{4-h}$ (quad. formula)

$(c-2)^2 = 4-h$

bifurcation point $h=4$

bifurcation diagram on right

Homework. 5-21 (odd)